Partial Categorical Multi-Combinators and Church-Rosser Theorems

Rafael Dueire Lins

[1]Universidade Federal de Pernambuco
DES - CTG, Recife, PE, Brazil

rdl@ee.ufpe.br

Abstract. *Categorical Multi-Combinators form a rewriting system developed with the aim of providing efficient implementations of lazy functional languages. This system allows the equivalent of several β-reductions to be performed at once, as functions form frames with all their arguments. Although this feature is convenient for most cases of function application it does not allow partially parameterised functions to fetch arguments. This paper presents Partial Categorical Multi-Combinators, a new rewriting system, which removes this drawback.*

Resumo. *Este artigo descreve o sistema de reescrita dos Multi-Combinadores Categóricos Parciais, que possui a propriedade de efetuar o equivalente a uma série de β reduções do λ-cálculo em um único passo de reescrita e possibilita aplicações parciais.*

Introduction

The method of compilation of functional languages into combinators, first explored by Turner in[20], provides a way of removing the variables from a program, transforming it into an applicative combination of constant functions or **combinators**. Turner used a set of combinators based on Curry's Combinatory Logic. To each combinator there is associated a rewriting law. In rewriting a combinator expression, Turner rewrites the leftmost-outermost reducible subexpression (or *redex*) at each stage. When no further rewriting can take place the expression is said to be in **normal form**.

Another theory of functions is provided by Category Theory[6], and we can see the notation used herein as providing an alternative set of combinators. The original system of Categorical Combinators was developed by Curien[2]. This work was inspired by the equivalence of the theories of typed λ-calculus and Cartesian Closed Categories as shown by Lambek[6] and Scott[19].

Aiming to implement lazy functional languages in an efficient way using rewriting of Categorical Combinators we developed a number of optimisations[7, 8] of the naïve system, the most refined of which was the system of Linear Categorical Combinators[8]. The modifications introduced reduce the number of rewriting laws and increase the efficiency of the system by reducing the number of rewriting steps involved in taking an expression to normal form, whilst leaving the complexity of the pattern matching algorithm unchanged.

Categorical Multi-Combinators are a generalisation of Linear Categorical Combinators. Each rewriting step of the Multi-Combinator code is equivalent to several rewritings of Linear Categorical Combinators, since an application of a function to several arguments can be reduced in a single step. The core of the system of Categorical Multi-Combinators consists only of four rewriting laws with a very low pattern-matching complexity and avoids the generation of trivially reducible sub-expressions. In [12] we have shown the equivalence between the operational semantics of the TIM [3] machine and rewriting of Categorical Multi-Combinator expressions: every TIM state is equivalent to a Categorical Multi-Combinator expression and *vice versa*; equivalent expressions are transformed into equivalent expressions by rewriting.

Independently, there has been much interest in compiled versions of functional languages which run much more quickly on von Neumann machines than do interpreters. Johnsson, with his implementation of Lazy ML[5], showed that it is possible to get fast implementations of lazy functional languages. The basic principle of the G-Machine is to avoid generating graphs. An analysis of the G-Machine and its optimisations can be found in [13]. Categorical Multi-Combinators served as a basis for several compiled machines [10, 11, 16, 17]. The latest one, ΓCMC [10], has already shown very good performance figures [18].

The system of Categorical Multi-Combinators allows the equivalent of several β-reductions to be performed at once, as functions form frames with all their arguments. This feature is convenient for most cases of function application because a coarser granularity of computation allows better compiled code. On the other hand, full lazyness is lost because partially parameterised functions are not reducible as such. Partial applications need to wait until the evaluation reaches a point in which all arguments are present, becoming a total application. If a partial application becomes shared in Categorical Multi-Combinators a copy of it is made for each instance of the variable to be replaced, losing the sharing of computations. Pseudoknot [18] is an example of a benchmark in which there is a large number of shared partial applications. In this paper we present a new set of Categorical Multi-Combinators, called Partial Categorical Multi-Combinators, which allows partial applications to be evaluated. We prove that Partial Categorical Multi-Combinators have the Church-Rosser properties of uniqueness of normal forms and that they are normalising, i.e. rewriting the leftmost-outermost pattern at each point of the reduction sequence leads to normal form, if it exists.

Categorical Multi-Combinators

In this section we present the compilation algorithm and rewriting laws for Categorical Multi-Combinators.

Compilation Algorithm

In Categorical Multi-Combinators, function application is denoted by juxtaposition, taken to be left-associative. The compilation algorithm for translating λ-expressions into Categorical Multi-Combinators is:

(T.1) $[\underbrace{\lambda x_i \ldots \lambda x_j}_{n}.a] = \langle L^{n-1}(R^{x_i \ldots x_j} a), ()\rangle$
(T.2) $[a \ldots b] = [a] \ldots [b]$
(T.3) $[c] = c$, where c is a constant.
(T.4) $R^{x_i \ldots x_j} \underbrace{\lambda x_k \ldots \lambda x_l}_{m}.a = L^{m-1}(R^{x_i \ldots x_j x_k \ldots x_l} a)$
(T.5) $R^{x_i \ldots x_j}(a \ldots b) = (R^{x_i \ldots x_j} a \ldots R^{x_i \ldots x_j} b)$
(T.6) $R^{x_i \ldots x_j} b = \begin{cases} b, \text{ if } b \text{ is a constant} \\ n_k, \text{if } b = x_k \end{cases}$

In the case of rules T.1 and T.4 above, n and m stand for the largest possible sequence of binders, i.e. a may not be an abstraction. $R^{x_i \ldots x_j}$ is an auxiliary function which at T.6 replaces variables by its deBruijn number, the depth in the list of bound variables generated by T.1 and expanded by T.4. Rule T.5 above, simply distributes the environment (list of bound variables) through applications. Whenever applying rule T.6 above a variable b can be associated with more than one x_k one must choose the minimum corresponding n_k, keeping locality of binding.

Categorical Multi-Combinator Rewriting Laws

The core of the Categorical Multi-Combinator machine is presented on page 71 of [9]. For a matter of convenience the multi-pair combinator, which forms evaluation environments, is written as $(x_0, \ldots, x_n)$. Compositions, which represent closures, are denoted as $\langle a, b\rangle$. Using this notation the kernel of the Categorical Multi-Combinator rewriting laws is expressed as:

(M*.1) $\langle n, (x_m, \ldots, x_1, x_0)\rangle \Rightarrow x_n$
(M*.2) $\langle x_0 x_1 x_2 \ldots x_n, y\rangle \Rightarrow \langle x_0, y\rangle \ldots \langle x_n, y\rangle$
(M*.3) $\langle L^n(y), (w_0, \ldots)\rangle x_0 x_1 \ldots x_n x_{n+1} \ldots x_z \Rightarrow \langle y, (x_0, \ldots, x_n)\rangle x_{n+1} \ldots x_z$
(M*.4) $\langle k, (x_m, \ldots, x_1, x_0)\rangle \Rightarrow k$, where k is a constant

The state of computation of a Categorical Multi-Combinator expression is represented by the expression itself. Rule (M*.1) performs environment look-up, this is the mechanism by which a variable fetches its value in the corresponding environment. (M*.2) is responsible for environment distribution. Rule (M*.3) performs environment formation. It is called multi-β reduction, because it is equivalent to performing several β-reductions in the λ-calculus. Rule (M*.4) discards the environment associated with a constant.

$(\lambda a.a\ \ a)((\lambda c \lambda d.d)B)C$ is translated into Categorical Multi-Combinators and rewritten as (we assume that $[B] = B'$ and $[C] = C'$),

$$\begin{aligned}
\langle L^0(0\,0), ()\rangle(\langle L^1(0), ()\rangle\ B')C' &\overset{M*.3}{\Rightarrow} \langle(0\,0), (\langle L^1(0), ()\rangle\ B')\rangle\ C' \\
&\overset{M*.2}{\Rightarrow} (\langle 0, (\langle L^1(0), ()\rangle\ B')\rangle\langle 0, (\langle L^1(0), ()\rangle\ B')\rangle)\ C' \\
&\overset{M*.1}{\Rightarrow} \langle L^1(0), ()\rangle\ B'\ \langle 0, (\langle L^1(0), ()\rangle\ B')\rangle)\ C' \\
&\overset{M*.3}{\Rightarrow} \langle 0, (B', \langle 0, (\langle L^1(0), ()\rangle\ B')\rangle)\rangle C' \\
&\overset{M*.1}{\Rightarrow} \langle 0, (\langle L^1(0), ()\rangle\ B')\rangle\ C' \\
&\overset{M*.1}{\Rightarrow} \langle L^1(0), ()\rangle\ B'\ C' \\
&\overset{M*.3}{\Rightarrow} \langle 0, (B', C')\rangle \overset{M*.1}{\Rightarrow} C'
\end{aligned}$$

Reduction Order

It is a well known fact that leftmost-outermost reduction of λ-expressions is a safe but non-optimal reduction strategy. In the λ-expression

$$(\lambda a.a\ a)((\lambda c \lambda d.d)B)\ C$$

the reduction of the rightmost redex yields

$$\begin{aligned} &\stackrel{\beta}{\Rightarrow} \ (\lambda a.a\ a)(\lambda d.d)\ C \\ &\stackrel{\beta}{\Rightarrow} \ (\lambda d.d)(\lambda d.d)\ C \\ &\stackrel{\beta}{\Rightarrow} \ (\lambda d.d)\ C \\ &\stackrel{\beta}{\Rightarrow} \ C \end{aligned}$$

The sequence of reductions above is shorter than the leftmost-outermost one, because the partial application, which forms the rightmost redex in the expression above, was reduced before being copied. Functional programs often make use of partially applied functions [18]. A program that makes intensive use of partial applications which become shared during execution calls for an efficient mechanism allowing the sharing of computation to be kept.

If we analyse the sequence of reductions for Categorical Multi-Combinators we can see that the Categorical Multi-Combinator sub-expression equivalent to the rightmost redex in the λ-expression is not reducible by applying any of the rewriting laws above. Categorical Multi-Combinators will make copies of the partial application and “wait” until all arguments are present to perform multi-β reduction. As functional languages only print expressions of ground type, we know that the extra arguments needed will be in place whenever the partial application becomes the leftmost-outermost redex, thus making multi-β reduction possible. However, not being able to share the result of evaluation of partial applications has performance implications.

Partial Categorical Multi-Combinators

In this section we introduce Partial Categorical Multi-Combinators, a rewriting system which allows to reduce partially applied functions.

Compilation Algorithm

The compilation algorithm for translating λ-expressions into Partial Categorical Multi-Combinators is different from that presented above for Categorical Multi-Combinators. Now, instead of working with the deBruijn representation for variables we work with the co-deBruijn number, as we want variables to which arguments are passed first to be represented by smaller numbers than the ones which correspond to arguments passed later on. Parenthesisation of expressions is also made explicit. Thus the compilation algorithm for Partial Categorical Multi-Combinators from fully parenthesised λ-lifted expressions in the λ-Calculus is:

(T'.1) $[\underbrace{\lambda x_i \ldots \lambda x_j}_{n} .a] = \langle L^{n-1}(R^{x_j \ldots x_i} a), ()\rangle$
(T'.2) $[(\ldots(a\, b)\ldots c)] = (\ldots([a][b])\ldots[c])$
(T'.3) $[c] = c$, where c is a constant.
(T'.4) $R^{x_j \ldots x_i} \underbrace{\lambda x_k \ldots \lambda x_l}_{m} .a = L^{m-1}(R^{x_l \ldots x_k x_j \ldots x_i} a)$
(T'.5) $R^{x_j \ldots x_i}(\ldots(a\, b)\ldots c) = (R^{x_j \ldots x_i} a \ldots R^{x_j \ldots x_i} b)\ldots R^{x_j \ldots x_i} c$
(T'.6) $R^{x_j \ldots x_i} b = \begin{cases} b, \text{ if } b \text{ is a constant} \\ n_k, \text{if } b = x_k \end{cases}$

The remarks made on the compilation algorithm for Categorical Multi-Combinators still hold. Again, if whenever applying rule T'.6 above a variable b can be associated with more than one x_k, then one must choose the maximum corresponding n_k. This enforces the locality of binding of variables. Observing the compilation algorithm above one can see that the only difference to the Categorical Multi-Combinators (rules T.1 to T.6) is the representation of variables by the co-deBruijn number.

Example of Compilation

Here follows an example of the compilation of a λ-expression into Partial Categorical Multi-Combinators, using the algorithm above:

$$\begin{aligned}
[(((\lambda a.a\ a)((\lambda c \lambda d.d)B))\ C)] &\stackrel{T'.2}{=} (([(\lambda a.a\ a)]\ [((\lambda c \lambda d.d)B)])\ [C]) \\
&\stackrel{T'.1}{=} ((\langle L^0(R^a(a\ a)), ()\rangle\ [((\lambda c \lambda d.d)B)])\ [C]) \\
&\stackrel{T'.5}{=} ((\langle L^0(R^a a\ R^a a), ()\rangle\ [((\lambda c \lambda d.d)B)])\ [C]) \\
&\stackrel{T'.6}{=} ((\langle L^0(0\ R^a a), ()\rangle\ [((\lambda c \lambda d.d)B)])\ [C]) \\
&\stackrel{T'.6}{=} ((\langle L^0(0\ 0), ()\rangle\ [((\lambda c \lambda d.d)B)])\ [C]) \\
&\stackrel{T'.2}{=} ((\langle L^0(0\ 0), ()\rangle\ ([\lambda c \lambda d.d]\ [B]))\ [C]) \\
&\stackrel{T'.1}{=} ((\langle L^0(0\ 0), ()\rangle\ (\langle L^1(R^{d,c} d), ()\rangle\ [B]))\ [C]) \\
&\stackrel{T'.6}{=} ((\langle L^0(0\ 0), ()\rangle\ (\langle L^1(1), ()\rangle\ [B]))\ [C])
\end{aligned}$$

The size of compiled expressions in Partial and Categorical Multi-Combinators is exactly the same and is linear with their λ-calculus equivalent.

Partial Categorical Multi-Combinators Rewriting Laws

In this section we generalise multi-β reduction to allow a function to fetch fewer arguments than its arity passed to it. Thus one has,

$$(\ldots(\langle L^n(y), (w_1, \ldots)\rangle)x_0)x_1)\ldots)x_m) \Rightarrow \langle L^{n-m-1}(\langle y, (x_0, \ldots, x_m)\rangle), ()\rangle \qquad if\, m < n$$

Now, one needs to adjust the argument fetching mechanism in such a way to allow variables to work with partial multi-β reduction.

$$\langle n, (x_m, \ldots, x_1, x_0)\rangle \Rightarrow \begin{cases} x_n, \text{ if } n \leq m \\ n - m - 1, \textit{otherwise} \end{cases}$$

The complete set of rewriting laws for Partial Categorial Multi-Combinators is:

(P.1) $\langle n, (x_m, \ldots, x_1, x_0)\rangle \Rightarrow \begin{cases} x_n, \text{ if } n \leq m \\ n - m - 1, \textit{otherwise} \end{cases}$

(P.2) $\langle x_0 x_1 x_2 \ldots x_n, y\rangle \Rightarrow \langle x_0, y\rangle \ldots \langle x_n, y\rangle$

(P.3) $(\ldots(\langle L^n(y), (w_1, \ldots)\rangle) x_0) \ldots) x_n) x_{n+1}) \ldots) x_z) \Rightarrow$
$(\ldots\langle y, (x_0, \ldots, x_n)\rangle), x_{n+1}) \ldots) x_z)$

(P.4) $(\ldots(\langle L^n(y), (w_1, \ldots)\rangle x_0) \ldots) x_m) \Rightarrow \langle L^{n-m-1}(\langle y, (x_0, \ldots, x_m)\rangle), ()\rangle \; if m < n$

(P.5) $\langle k, (x_m, \ldots, x_1, x_0)\rangle \Rightarrow k,$ where k is a constant

The fundamental difference between Partial and Categorical Multi-Combinators above is rewriting law P.4 above. It allows a function with less arguments than its arity to process the existing arguments yielding another function on the remaining arguments. Law P.4 restores an adequate degree of currying to the system of Categorical Multi-Combinators lost by λ-lifting, without incurring the penalty of having redundant laziness.

Example of Evaluation

Let us analyse the Partial Categorical Multi-Combinator expression presented in the example above under a reduction strategy similar to the one adopted for the reduction of the λ-expression in the last section, i.e. reducing the rightmost redex first.

$$
\begin{aligned}
(((\langle L^0(0\,0), ()\rangle\, (\langle L^1(1), ()\rangle\, B'))\, C') &\overset{P.4}{\Rightarrow} (((\langle L^0(0\,0), ()\rangle\, (\langle L^0(\langle 1, B'\rangle\,), ()\rangle)\, C') \\
&\overset{P.1}{\Rightarrow} (((\langle L^0(0\,0), ()\rangle\, \langle L^0(0), ()\rangle\,)\, C')
\end{aligned}
$$

at this point the partial parameterisation of the function on the right hand side of the expression above was fully reduced, giving rise to a new function. Evaluation proceeds as follows:

$$
\begin{aligned}
&\overset{P.3}{\Rightarrow} (\langle(0\,0), (\langle L^0(0), ()\rangle\,)\rangle\, C') \\
&\overset{P.2}{\Rightarrow} ((\langle 0, (\langle L^0(0), ()\rangle\,)\rangle)\, \langle 0, (\langle L^0(0), ()\rangle)\rangle\, C') \\
&\overset{P.1}{\Rightarrow} (\langle L^0(0), ()\rangle\;\; \langle 0, (\langle L^0(0), ()\rangle)\rangle\, C') \\
&\overset{P.3}{\Rightarrow} (\langle 0, (\langle\, 0, (\langle L^0(0), ()\rangle)\rangle)\rangle\, C') \\
&\overset{P.1}{\Rightarrow} (\langle 0, (\langle L^0(0), ()\rangle)\rangle)\, C') \\
&\overset{P.1}{\Rightarrow} (\langle L^0(0), ()\rangle)))\, C') \\
&\overset{P.3}{\Rightarrow} \langle 0, (C')\rangle \\
&\overset{P.1}{\Rightarrow} C'
\end{aligned}
$$

Below, we prove that Partial Categorical Multi-Combinators have the Church-Rosser property allowing rewritings to take place in any order to reach normal form, if it exists. Notice that applicative order in Categorical Multi-Combinators yields expressions equivalent to λ-expressions in *head-normal forms*. Applicative order reduction of Partial Multi-Combinator expressions yields expressions equivalent to expressions in *normal* form in the λ-Calculus.

Church-Rosser Theorems

The first Church-Rosser theorem for the λ-Calculus proves the uniqueness of normal forms of λ-expressions, if they exist. This means that all terminating sequences of reductions of a λ-expression will lead to the same result. A rewriting system to which the Church-Rosser property is valid is called *confluent* or Church-Rosser. The second Church-Rosser theorem for the λ-Calculus shows that the reduction of the leftmost-outermost redex at each point of the reduction sequence leads to normal form, if it exists.

In this section, we show that Partial Categorical Multi-Combinators have the properties stated by the two Church-Rosser theorems.

Normal Forms

Here we prove that Partial Categorical Multi-Combinators have the property that if one starts from a Partial Categorical Multi-Combinators expression any terminating sequence of reductions leads to the same expression.

Our strategy for proving this property is based on Huet's version of the Knuth-Bendix algorithm [4]. Huet proves that if a rewriting system if *left-linear* and has no *critical pairs* it is confluent. A rewriting system is said to be left-linear if no variable appears more than once on the left-hand side of any of its rewriting rules. Critical pairs are computed by a superposition algorithm, where one attempts to match in a most general way the left-hand side of some rewriting rule with a nonvariable subterm of all rewriting rules in the system, including itself. Critical pairs show the possibility of reduction sequences diverging. Huet's result is easily extensible to a conditional rewriting system where mutually exclusive clauses do not give rise to critical pairs.

The analysis of the set of rewriting laws for Partial Categorical Multi-Combinators shows that there is no repeated variable on the left-hand side of any of the rewriting rules. Considering that rules P.3 and P.4 are mutually exclusive, there is no possible overlapping of patterns on the left hand side of any of the rewriting laws. Any rewritable pattern matches trivially with a variable of any of the rewriting laws in the system, therefore there are no critical pairs. We have proved that Partial Categorical Multi-Combinators form a confluent rewriting system, thus the Church-Rosser property of uniqueness of normal forms holds.

Normalisation Property

This section presents the proof that the reduction of the leftmost-outermost redex at each point of the reduction sequence leads to normal form, if it exists. A direct proof of this theorem is not simple. Our strategy is to produce a proof in three steps. First, we present the λ-Calculus with lazy substitutions [7], a rewriting system which performs β-reductions with explicit, on demand, variable substitution. The second step is to introduce the λ-Calculus with Multiple Substitutions, a rewriting system in which each rewriting step is equivalent to several β-reductions. Variable substitution is also performed explicitly and on demand. Then, we show that leftmost-outermost rewritings of Partial Categorical Multi-Combinators are equivalent to leftmost-outermost rewritings on the λ-Calculus with Multiple Substitutions, therefore equivalent in each step to a sequence of leftmost-outermost β-reductions on the λ-Calculus.

The λ-Calculus with Lazy Substitutions

The rewriting system called the λ-Calculus with lazy substitutions was introduced in [7] as a way to prove that the leftmost-outermost rewriting of Simplified Categorical Combinators [8] was equivalent to performing leftmost-outermost β-reductions in the λ-Calculus.

The rewriting laws in the λ-Calculus with lazy substitutions are:

l.1 $(\lambda x.a)A \Rightarrow [A/x]a$

l.2 $[A/x]\lambda z.a \Rightarrow \begin{cases} \lambda z.a, & if\, x \neq z \\ \lambda z.[A/x]a, & if\, x \neq z,\, and\ z\ not\ free\ in\ A \\ \lambda z.[A/x][w/z]a, & where\ w\ is\ a\ new\ variable \end{cases}$

l.3 $[A/x]\ (a\ b) \Rightarrow ([A/x]a)\ ([A/x]b)$

l.4 $[A/x]\ x \Rightarrow A$

l.5 $[A/x]\ z \Rightarrow z$

Rule 1.1 above is β-reduction with an explicit variable substitution operator $[A/x]$. Rules l.2 and l.3 shifts the substitution operator into the body of an abstraction and distributes it through an application, respectively. Rules l.4 and l.5 perform actual substitution of formal parameters for real parameters. It is obvious that the leftmost-outermost reduction in the λ-Calculus with lazy substitutions is equivalent to leftmost-outermost β-reduction in the λ-Calculus.

The λ-Calculus with Multiple Substitutions

Assuming we have the following λ-expression,

$$(\lambda x.\lambda y.\lambda z.a)\ T\ U\ V\ \ldots$$

applying the rules of the λ-Calculus with lazy substitutions it leftmost-outermost reduces to:

$$\begin{array}{ll} \overset{\Rightarrow}{l.1} & ([T/x]\lambda y.\lambda z.a)\ U\ V\ \ldots \\ \overset{\Rightarrow}{l.2} & (\lambda y.[T/x]\lambda z.a)\ U\ V\ \ldots \\ \overset{\Rightarrow}{l.1} & ([U/y][T/x]\lambda z.a)\ V\ \ldots \\ \overset{\Rightarrow}{l.2} & ([U/y]\lambda z.[T/x]a)\ V\ \ldots \\ \overset{\Rightarrow}{l.2} & (\lambda z.[U/y][T/x]a)\ V\ \ldots \\ \overset{\Rightarrow}{l.1} & ([V/z][U/y][T/x]a)\ldots \end{array}$$

One can observe in the sequence of reductions above that no other rewriting takes place until all the substitution operators appear. There is no reason for not rewriting the top expression directly into the bottom one, as this is always the leftmost-outermost rewriting path, yielding:

$$(\lambda x.\lambda y.\lambda z.a)\ T\ U\ V\ \ldots \Rightarrow ([V/z][U/y][T/x]a)\ldots$$

Making this a new rewriting law and adopting a more convenient notation for the substitution operator we present a new rewriting system called the λ-Calculus with Multiple Substitutions:

m.1 $(\lambda x_1.\lambda x_2.\ldots\lambda x_n.a)A_1\ldots A_m \Rightarrow \lambda x_{m+1}\ldots\lambda x_n.[A_1/x_1,\ldots,A_m/x_m]a,\quad$ if $m<n$.
m.2 $(\lambda x_1.\lambda x_2.\ldots\lambda x_n.a)A_1\ldots A_nA_{n+1}\ldots \Rightarrow [A_1/x_1,\ldots,A_n/x_n]aA_{n+1}\ldots,$ otherwise.
m.3 $[A_1/x_1,\ldots,A_n/x_n]\lambda z.a \Rightarrow \lambda z.[A_1/x_1,\ldots,A_n/x_n]a$
m.4 $[A_1/x_1,\ldots,A_n/x_n](a\ldots b) \Rightarrow ([A_1/x_1,\ldots,A_n/x_n]a\ldots[A_1/x_1,\ldots,A_n/x_n]b)$
m.5 $[A_1/x_1,\ldots,A_n/x_n]x_i \Rightarrow A_i$
m.6 $[A_1/x_1,\ldots,A_n/x_n]z \Rightarrow z$

In rule m.3 we assume that for all i we have $x_i \neq z$ and that z does not appear free in any expression A_i, α-conversion may be needed to guarantee this condition.

One can see by construction that the leftmost-outermost reduction in the λ-Calculus with Multiple Substitutions is equivalent to a sequence of leftmost-outermost reductions in the λ-Calculus with Lazy Substitutions, therefore any terminating sequence in the former gives rise to one in the latter; thus this reduction strategy is normalising.

Final Step

Now we prove that the rewriting of the leftmost-outermost pattern of Partial Categorical Multi-Combinators corresponds to rewriting the leftmost-outermost redex in the λ-Calculus with Multiple Substitutions. We first introduce a translation function $\mathcal{T}$, which translates Partial Categorical Multi-Combinator expressions into expressions of the λ-Calculus with Multiple Substitutions. The translation function $\mathcal{T}$ is defined as:

(t.1) $\mathcal{T}^{w_m\ldots w_1}(\ldots(a\ b)\ldots)l) = \mathcal{T}^{w_m\ldots w_1}a\mathcal{T}^{w_m\ldots w_1}b\ldots\mathcal{T}^{w_m\ldots w_1}l$
(t.2) $\mathcal{T}^{w_m\ldots w_1}L^n(\langle y,(x_1,\ldots,x_m)\rangle) = (\lambda w_{m+1}\ldots\lambda w_{n-m+1}.\mathcal{T}^{w_{m+n+1}\ldots w_1}\langle y,(x_0,\ldots,x_m)\rangle)$
(t.3) $\mathcal{T}^{w_m\ldots w_1}\langle L^n(y),(x_m,\ldots,x_1)\rangle) = (\lambda w_1\ldots\lambda w_{m+1}.\mathcal{T}^{w_{m+n}\ldots w_1}y)$
(t.4) $\mathcal{T}^{w_m\ldots w_1}\langle y,(x_n,\ldots,x_1)\rangle = [\mathcal{T}^{\Box}x_1/w_1,\ldots,\mathcal{T}^{\Box}x_n/w_n]\mathcal{T}^{w_{m+n}\ldots w_1}y$
(t.5) $\mathcal{T}^{w_m\ldots w_1}n = w_{n+1},\qquad$ if n is a variable.
(t.6) $\mathcal{T}^{w_m\ldots w_1}k = k,\qquad$ if k is a constant.

One can observe that the translation function $\mathcal{T}$ is a correct mapping between the two rewriting systems by analysing the behaviour of original and translated expressions. One can see that $\mathcal{T}$ and $\mathcal{R}$ behave almost as inverses of each other. We show that if a Partial Categorical Multi-Combinator expression A leftmost-outermost rewrites in one step to an expression A', then the translation of A into the λ-Calculus with Multiple Substitutions, $\mathcal{T}\ A$, leftmost-outermost rewrites to $\mathcal{T}\ A'$, in one step. So, the following diagram commutes:

$$\begin{array}{ccc} A & \stackrel{\mathcal{T}}{\to} & \mathcal{T}\ A \\ \Downarrow & & \Downarrow \\ A' & \stackrel{\mathcal{T}}{\to} & \mathcal{T}\ A' \end{array}$$

We will analyse each of the rewritable patterns for Partial Categorical Multi-Combinators.

P.1

$$\begin{array}{rcl} \mathcal{T}^{\Box}\langle n,(x_0,\ldots,x_m)\rangle & \stackrel{t.4}{=} & [\mathcal{T}^{\Box}x_0/w_1,\ldots,\mathcal{T}^{\Box}x_m/w_{m+1}]\mathcal{T}^{w_{m+1}\ldots w_1}n \\ \Downarrow P.1 & & \Downarrow t.5 \\ \mathcal{T}^{\Box}x_n & & [\mathcal{T}^{\Box}x_0/w_1,\ldots,\mathcal{T}^{\Box}x_m/w_{m+1}]w_{n+1} \\ & & \Downarrow m.5 \\ & & \mathcal{T}^{\Box}x_n \end{array}$$

The second clause in rule P.1 is never the leftmost-outermost redex in a Partial Multi-Combinator expression. If it were, there would be a situation equivalent to existing a free variable in the code.

P.2

$$\mathcal{T}^{\Box}(\langle(x_0,\ldots,x_n),(v_1,\ldots,v_m)\rangle) \quad \overset{t.4}{=}$$
$$\Downarrow P.2$$
$$\mathcal{T}^{\Box}(\langle x_0,(v_1,\ldots,v_m)\rangle)\ldots(\langle x_n,(v_1,\ldots,v_m)\rangle)$$
$$\Downarrow t.1$$
$$\mathcal{T}^{\Box}(\langle x_0,(v_1,\ldots,v_m)\rangle)\ldots\mathcal{T}^{\Box}(\langle x_n,(v_1,\ldots,v_m)\rangle)$$
$$\Downarrow t.4$$
$$[\mathcal{T}^{\Box}v_1/w_1,\ldots]\mathcal{T}^{w_m\cdots}x_0\ldots[\mathcal{T}^{\Box}v_1/w_1,\ldots]\mathcal{T}^{w_m\cdots}x_n$$

$$\overset{t.4}{=} \quad [\mathcal{T}^{\Box}v_1/w_1,\ldots,\mathcal{T}^{\Box}v_m/w_m]\mathcal{T}^{w_m\cdots}(x_0,\ldots,x_n)$$
$$\Downarrow t.1$$
$$[\mathcal{T}^{\Box}v_1/w_1,\ldots]\mathcal{T}^{w_m\cdots}x_0\ldots\mathcal{T}^{w_m\cdots}x_n$$
$$\Downarrow m.4$$
$$[\mathcal{T}^{\Box}v_1/w_1,\ldots]\mathcal{T}^{w_m\cdots}x_0\ldots[\mathcal{T}^{\Box}v_1/w_1,\ldots]\mathcal{T}^{w_m\cdots}x_n$$

P.3

$$\mathcal{T}^{\Box}(\ldots(\langle L^n(y),(v_l,\ldots,v_1)\rangle x_0)\ldots)x_n)\ldots)x_z) \quad \overset{=}{t.1}$$
$$\Downarrow P.3$$
$$\mathcal{T}^{\Box}(\ldots(\langle y,(x_0,\ldots,x_n)\rangle x_{n+1})\ldots)x_z)$$
$$\Downarrow t.1$$
$$\mathcal{T}^{\Box}(\ldots(\langle y,(x_0,\ldots,x_n)\rangle\ \mathcal{T}^{\Box}x_{n+1}\ldots\ \mathcal{T}^{\Box}x_z$$
$$\Downarrow t.4$$
$$[\mathcal{T}^{\Box}(x_0/w_1,\ldots,\mathcal{T}^{\Box}x_n/w_{n+1})]\ \mathcal{T}^{w_{n+1}\ldots w_1}y\mathcal{T}^{\Box}x_{n+1}\ldots\ \mathcal{T}^{\Box}x_z$$

$$\overset{=}{t.1} \quad \mathcal{T}^{\Box}(\ldots(\langle L^n(y),(v_l,\ldots,v_1)\rangle\mathcal{T}^{\Box}x_0\ldots\ \mathcal{T}^{\Box}x_n\ldots\mathcal{T}^{\Box}x_z$$
$$\Downarrow t.3$$
$$(\lambda w_1\ldots\lambda w_{n+1}.\mathcal{T}^{w_{n+1}\ldots w_1}y)\mathcal{T}^{\Box}x_0\ldots\ \mathcal{T}^{\Box}x_n\ldots\mathcal{T}^{\Box}x_z$$
$$\Downarrow m.2$$
$$[\mathcal{T}^{\Box}(x_0/w_1,\ldots,\mathcal{T}^{\Box}x_n/w_{n+1})]\ \mathcal{T}^{w_{n+1}\ldots w_1}y\mathcal{T}^{\Box}x_{n+1}\ldots\ \mathcal{T}^{\Box}x_z$$

P.4

$$
\begin{array}{rl}
\mathcal{T}^{\square}(\ldots(\langle L^n(y),(v_l,\ldots,v_1)\rangle x_0)\ldots)x_m) & \bar{\bar{t}}.1 \\
\Downarrow P.4 & \\
\mathcal{T}^{\square}(\ldots(L^{n-m-1}(\langle y,(x_0,\ldots,x_m)\rangle) & \\
|| t.2 & \\
(\lambda w_{m+1}\ldots\lambda w_{n-m-1}.\mathcal{T}^{w_{n-m}\ldots w_1}\langle y,(x_0,\ldots,x_m)\rangle) & \\
|| t.4 & \\
(\lambda w_1\ldots\lambda w_{n-m}.[\mathcal{T}^{\square}x_0/w_1,\ldots,\mathcal{T}^{\square}x_m/w_{m+1}]\ \mathcal{T}^{w_{n+1}\ldots w_1}y) &
\end{array}
$$

$$
\begin{array}{ll}
\bar{\bar{t}}.1 & \mathcal{T}^{\square}(\ldots(\langle L^n(y),(v_l,\ldots,v_1)\rangle\mathcal{T}^{\square}x_0\ldots\ \mathcal{T}^{\square}x_m \\
& \quad\Downarrow t.3 \\
& (\lambda w_1\ldots\lambda w_{n+1}.\mathcal{T}^{w_{n+1}\ldots w_1}y)\mathcal{T}^{\square}x_0\ldots\ \mathcal{T}^{\square}x_m \\
& \quad\Downarrow m.1 \\
& (\lambda w_1\ldots\lambda w_{n-m}.[\mathcal{T}^{\square}x_0/w_1,\ldots,\mathcal{T}^{\square}x_m/w_{m+1}]\ \mathcal{T}^{w_{n+1}\ldots w_1}y)
\end{array}
$$

P.5

$$
\begin{array}{ccl}
\mathcal{T}^{\square}\langle k,(x_1,\ldots,x_m)\rangle & \bar{\bar{t}}.4 & [\mathcal{T}^{\square}x_1/w_1,\ldots,\mathcal{T}^{\square}x_m/w_m]\ \mathcal{T}^{w_m\ldots w_1}k \\
\Downarrow P.5 & & \quad || t.6 \\
\mathcal{T}^{\square}k & \bar{\bar{t}}.4 & [\mathcal{T}^{\square}x_1/w_1,\ldots,\mathcal{T}^{\square}x_m/w_m]\ k \\
|| t.6 & & \quad\Downarrow m.6 \\
k & & \quad k
\end{array}
$$

Because $\mathcal{T}$ makes no redex transposition during translation, the leftmost-outermost Partial Categorical Multi-Combinator redex corresponds to the leftmost-outermost redex in the λ-Calculus with Multiple Substitutions, in the sense that the rewriting of the leftmost-outermost redexes in both systems are equal modulo translation. We can conclude that leftmost-outermost rewritings of Partial Categorical Multi-Combinators is a normalising strategy.

Conclusions

Partial Categorical Multi-Combinators form a rewriting system, which performs the equivalent to a sequence of β-reductions in one rewriting step, but also allow for the reduction of partially parameterised functions. This paper shows that Partial Categorical Multi-Combinators have the Church-Rosser properties of uniqueness of normal forms and that the rewriting of the leftmost-outermost pattern leads to normal form, if it exists. The introduction of Partial Categorical Multi-Combinator to ΓCMC brought full lazyness to the machine, allowing for partial applications to be shared. This strategy has proved efficient in our implementation of Haskell [1, 15], yielding a performance improvement of about 20% for the Pseudoknot benchmark [18].

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